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Direct and converse results for the weighted rational approximation of functions with inner singularities

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Abstract

The weighted rational approximation of functions with inner singularities of algebraic type in $[-1,1]$ is investigated. New direct and converse results not achievable by polynomials, are proved.

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Keywords: Inner singularities of algebraic type; Weighted modulus of smoothness; Weighted K-functional; Shepard-type operators; Converse results

1. Introduction

Recently in [7] the author studied the weighted uniform approximation of functions with algebraic singularities at the endpoints in $[-1,1]$ by rational interpolatory operators and direct and converse results not achievable by polynomials, were proved.

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The behaviour of the weighted approximation of functions with inner singularities by such rational operators was an open problem. Indeed, inner singularities add new difficulties and the behaviour of the approximation requires a more careful examination than in the case of endpoints singularities (cf. [12]). Recently in [4,12,13] the authors obtained error estimates for the best weighted polynomial approximation of functions with inner singularities of algebraic type in terms of a suitable weighted modulus of smoothness $\omega^{\varphi}(f;t)_w^*$, with $\varphi(x) = \sqrt{1-x^2}$.

The purpose of this paper is to construct rational operators for the weighted approximation of functions with inner singularities in $[-1,1]$. Such operators are positive, interpolatory and easy to construct. First we show that for such operators the weighted convergence with exponential type weights is not guaranteed in general (see Proposition 1). Therefore here we consider weights vanishing algebraically at any inner point and functions belonging to the class C_w defined in (2). For such functions we give convergence theorems and uniform approximation error estimates (see Theorem 2). We also establish the first pointwise approximation error estimate and the first converse results for functions from C_w (see Theorems 2 and 4). Our results involve a new weighted modulus of smoothness $\omega^{\varphi}(f)_w$ (see (7)) and an equivalent weighted K-functional $K^{\varphi}(f)_w$ (see Lemma 8) with $\varphi(x)$ vanishing algebraically at any inner point. This modulus is new, in a certain sense, because classical weighted moduli of smoothness involve functions having zeros at the endpoints of the approximation interval (cf. [11–13]). This peculiarity agrees with the different behaviour of polynomial and rational approximation. Useful tools for our results are some new weighted Markov–Bernstein type inequalities for our operator (see Lemmas 6 and 7) and the equivalence relation between our weighted modulus of smoothness and the corresponding weighted K-functional (see Lemma 8). Finally we prove that our estimates cannot be reached by polynomials (see Remarks to Theorem 2).

2. Main results

Letting

$$w(x) = |x-c|^{\alpha}, \quad \alpha > 0, \quad |c| < 1, \quad |x| \leqslant 1, \tag{1}$$

we introduce the following class of functions

$$C_w = \left\{ f \in C([-1,1] - \{c\}), \text{ s.t. } \lim_{x \to c^{\pm}} (wf)(x) = 0 \right\}. \tag{2}$$

Without loss of generality, we may assume $c = 0$ in (1), i.e., we consider functions having an inner singularity of algebraic type at 0. The case of several inner singularities can be similarly treated. For $f \in C_w$ put $\|wf\| := \sup_{|x| \leqslant 1} |(wf)(x)|$ and $\|wf\|_{[a,b]} := \sup_{x \in [a,b]} |(wf)(x)|$.

For odd n, construct the following mesh of nodes

$$x_{k,n} \equiv x_k = \begin{cases} -\left(1 - 2\frac{k}{n}\right)^{\beta}, & k = 0, \ldots, \frac{n-1}{2}, \\ \left(2\frac{k}{n} - 1\right)^{\beta}, & k = \frac{n+1}{2}, \ldots, n, \end{cases} \tag{3}$$

with $\beta \geqslant 1$ (cf. [8]). Note that this mesh is finer near $\pm 1/n^{\beta}$, ($\beta > 1$). Then consider the Shepard-type operator

$$S_n(f;x) = \sum_{k=0}^{n} f(x_k) \frac{|x - x_k|^{-s}}{\sum_{i=0}^{n} |x - x_i|^{-s}}, \tag{4}$$

with $|x| \leqslant 1$, $f \in C_w$ and $s > 2$. From the definition it follows that the operator S_n is positive, preserves constants, and interpolates to f at x_k, ($k = 0, \ldots, n$), in the sense that $\lim_{x \to x_k} S_n(f;x) = f(x_k)$ (cf. [1]). In the last decades the operator S_n and its multivariate extensions have been the subject of several papers, because of their interesting properties in classical approximation theory and other fields, as multivariate interpolation on scattered data, CAGD, fluid-dynamics (see, for example, [1–3,6,8–10,14–16]). Algorithms for parallel, multistage and recursive computation of S_n can be found in [1].

Here we first show that we have to consider weights of type (1) for the weighted approximation of $f \in C_w$ by S_n, since for exponential type weights the convergence is not guaranteed in general. Indeed we have

Proposition 1. Let $W(x) = \exp(-1/|x|)$ and $f(x) = \exp(1/\sqrt{|x|})$. Then

$$\limsup_n \| W S_n(f) \| = +\infty.$$

In the present paper we want to show that the operator defined by (3) and (4) is a good tool for the weighted uniform approximation of functions from C_w. In order to define the new modulus of smoothness we follow [4,12,13]. We consider the following main part of a weighted modulus of smoothness $\Omega^{\varphi}(f)_w$ defined by

$$\Omega^{\varphi}(f;t)_w = \sup_{0 < h \leqslant t} ||w(x)\left| f\left(x - h\frac{\varphi(x)}{2}\right) - f\left(x + h\frac{\varphi(x)}{2}\right)\right| ||_{I_h}, \tag{5}$$

where t is small (say $t < t_0$), $I_h = \left[-1 + h/2, -h^{\beta}\right] \cup \left[h^{\beta}, 1 - h/2\right]$ and $\varphi(x) = |x|^{(\beta-1)/\beta}$ (cf. [12] or [13], where $\varphi(x) = \sqrt{1 - x^2}$). If in (5) we put $\beta = 1$, i.e. $\varphi(x) = 1$, then

$$\begin{aligned} \Omega^1(f;t)_w &:= \Omega(f;t)_w \\ &= \sup_{0 < h \leqslant t} |||w(x)|f\left(x - \frac{h}{2}\right) - f\left(x + \frac{h}{2}\right)|||_{[-1+h/2,-h] \cup [h,1-h/2]}, \end{aligned} \tag{6}$$

is the main part of a weighted modulus of continuity.

In the following C will always denote a positive constant which, however, may assume different values in different occurrences. Moreover we write $a \sim b$, if $|a/b|^{\pm 1} \leqslant C$. Then we define the new weighted modulus of smoothness $\omega^\varphi(f;t)_w$ as

$$\omega^\varphi(f;t)_w = \Omega^\varphi(f;t)_w + \inf_{a \in \mathbb{R}} ||w[f-a]||_{J_t}, \tag{7}$$

with $J_t = [-t^\beta, t^\beta]$. For a discussion on definition (7) the reader can refer [4] or [5] or [12]. It is possible to prove that

$$\lim_{t \to 0^+} \omega^\varphi(f;t)_w = 0$$

and

$$\omega^\varphi(f;\mu t)_w \leqslant C(\mu+1)\omega^\varphi(f;t)_w, \ \forall \mu > 0, \tag{8}$$

with $C > 0$ depending only on w.

Then we state the following direct and converse results.

Theorem 2. *Let $f \in C_w$ and let S_n be the operator defined by* (3) *and* (4). *If $s \geqslant \alpha\beta + \beta + 1$, then*

$$||w[f - S_n(f)]|| \leqslant C\omega^\varphi\left(f;\frac{1}{n}\right)_w, \tag{9}$$

where $C > 0$ is a constant independent of f and n. Moreover, if $\alpha\beta \geqslant 1$, then

$$\omega^\varphi\left(f;\frac{1}{n}\right)_w + \frac{||wf||}{n} \sim ||w[f - S_n(f)]|| + \frac{1}{n}||w\varphi S_n'(f)|| + \frac{||wf||}{n}, \tag{10}$$

$$||w[f - S_n(f)]|| = O(n^{-\sigma}) \Leftrightarrow \omega^\varphi(f;t)_w = O(t^\sigma), \quad 0 < \sigma < 1, \tag{11}$$

and

$$||w\varphi S_n'(f)|| \leqslant Cn^{-\sigma+1} \Leftrightarrow \omega^\varphi(f;t)_w = O(t^\sigma), \quad 0 < \sigma \leqslant 1. \tag{12}$$

From (9) we deduce the weighted uniform convergence of $S_n(f)$ to f, $\forall f \in C_w$ and $\forall s \geqslant \alpha\beta + \beta + 1$. Our results are strongly influenced by the mesh distribution (see the function φ on the right-hand side in (9)–(12)).

We remark that estimate (9) cannot be obtained by polynomials. Indeed we have the following

Proposition 3. *For any $f \in C_w$ and every n there not exist an algebraic polynomial of degree at most n s.t.*

$$w(x)|f(x) - p_n(x)| \leqslant C\omega^\varphi\left(f;\frac{1}{n}\right)_w, \quad \forall |x| \leqslant 1. \tag{13}$$

From (10) by (9) we deduce (see (42))

$$||w\varphi S_n'(f)|| \leqslant C\left[n\omega^{\varphi}\left(f;\frac{1}{n}\right)_w + \frac{||wf||}{n}\right],$$

which is the analogous of [11, formula (7.3.1), p. 84] for the best approximation polynomial. From (10), by Lemma 8, it follows that

$$\inf_{||wh'\varphi||<\infty}\left\{||w[f-h]|| + \frac{1}{n}||w\varphi h'||\right\} + \frac{||wf||}{n} \sim + ||[f - S_n(f)]||\frac{1}{n}||w\varphi S_n'(f)|| + \frac{||wf||}{n}, \tag{14}$$

in other words the infimum on the right-hand side in (14) is essentially realized by $S_n(f)$. Moreover (9) cannot be improved because of (11). In a sense, equivalence relation (11) characterizes the class of functions from C_w having a given behaviour near 0 by the order of weighted approximation by operator S_n. Equivalence (12) is the analogous of the result in [11, Corollary 7.3, p. 86] for the best approximation polynomial.

We remark that (11)–(12) are the first converse results for functions from C_w.

We can also get a new pointwise approximation error estimate. Indeed as above construct the following mesh for odd n

$$y_k = \begin{cases} \left(\frac{2k}{n}\right)^{\beta}\left(1-\frac{1}{n}\right)^{1-\beta} + \frac{1}{n}, & k = 0, \ldots, \frac{n-1}{2}, \\ -y_{n-k}, & k = \frac{n+1}{2}, \ldots, n, \end{cases} \tag{15}$$

with $\beta > 1$. Note that this mesh is finer near $\pm\frac{1}{n}$. Then denote by $\overline{S}_n$ the operator S_n based on the new mesh (15). Moreover put $\epsilon_f(y) = \sup_{0<|x|\leqslant y} w(x)|f(x)|$, measuring the decay of $|(wf)(y)|$ to 0, when $|y| \to 0^+$ (cf. [9]).

Theorem 4. *Let* $s > \max\{\alpha\beta + 1, 1 + \beta/(\beta - 1)\}$. *Then*

$$w(x)|f(x) - \overline{S}_n(f;x)| \leqslant C\begin{cases} \epsilon_f(x) + \frac{w(x)}{w(y_0)}\epsilon_f(\mu_n) + \frac{w(x)}{w(\mu_n)}\epsilon_f(1), & \text{if } |x| < y_0, \\ \Omega\left(f;\frac{\varphi(x)}{n}\right)_w + \frac{\epsilon_f(2y_0+x)}{n^{s(1-1/\beta)-1}}, & \text{if } |x| > y_0, \end{cases} \tag{16}$$

where C *is a positive constant independent of* f *and* n, $\Omega(f;t)_w$ *is as in* (6), $\mu_n = Cn^{-\delta}, (0 < \delta < 1)$, *and* $\varphi(x) = |x|^{(\beta-1)/\beta}$

We observe that (16) is the first pointwise error estimate for functions from C_w.

3. Proofs of main results

Proof of Proposition 1. Put $x = (x_n + x_{n-1})/2$. Then

$$W(x)S_n(f;x) = \frac{\exp(-1/x)\sum_{k=0}^{n}|x-x_k|^{-s}\exp(1/\sqrt{|x_k|})}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$
$$\geqslant C\frac{\exp\left(1/\sqrt{|x_{[n/2]}|}\right)(x-x_{[n/2]})^{-s}}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$
$$\geqslant C\frac{\exp\left(Cn^{\beta/2}\right)}{\sum\limits_{k=0}^{n}|x-x_k|^{-s}}. \tag{17}$$

Since

$$\sum_{k=0}^{n}|x-x_k|^{-s}\leqslant Cn^s,$$

from (17) it follows that

$$W(x)S_n(f;x)\geqslant C\frac{\exp(Cn^{\beta/2})}{n^s},$$

which is unbounded when $n\to\infty$.

The proof of Theorem 2 is based on some preliminaries that are interesting in themselves since they establish the weighted boundedness of the operator S_n (Lemma 5), some new weighted Markov–Bernstein inequalities for S_n (Lemmas 6 and 7) and the equivalence relation between the weighted modulus of smoothness and the corresponding weighted K-functional (Lemma 8).

Lemma 5. *Let $s\geqslant\alpha\beta+1$. Then for every function f defined on $[-1,1]-\{0\}$ we have*

$$||wS_n(f)||\leqslant C||wf||,$$

where C is a positive constant independent of f and n.

Note that Lemma 5 does not use the assumption $\lim_{x\to 0}(wf)(x)=0$.

Lemma 6. If $s\geqslant\alpha\beta+1$, *then*

$$||w\varphi S_n'(f)||\leqslant Cn||wf||,$$

where C is independent of f and n.

The proofs of Lemmas 5 and 6 are similar to the proofs of Lemmas 3.1 and 3.2 in [9] (see also [7]) so we may omit them.

Lemma 7. *If* $s \geqslant \alpha\beta + 1$ *and* $\alpha\beta \geqslant 1$, *then*

$$||w\varphi S_n'(f)|| \leqslant C\{||w\varphi f'|| + ||wf||\}, \text{ if } ||w\varphi f'|| < C,$$

where C *is independent of* f *and* n.

Proof. Since

$$S_n(f;x) = f(x) + \sum_{k=0}^{n} A_k(x)[f(x_k) - f(x)],$$

with $A_k(x) = \dfrac{|x - x_k|^{-s}}{\sum_{i=0}^{n} |x - x_i|^{-s}}$, it follows that

$$S_n'(f;x) = \sum_{k=0}^{n} A_k'(x)[f(x_k) - f(x)].$$

Let $x > 0$ (the case $x < 0$ is similar). If $x_k > 0$, then

$$|(w\varphi)(x)| \left| \sum_{x_k > 0} A_k'(x)[f(x) - f(x_k)] \right| \leqslant |(w\varphi)(x)| \left| \sum_{x_k > 0} A_k'(x) \int_{x_k}^{x} f'(t)\,\mathrm{d}t \right| \tag{18}$$

and working as usual (cf. [6,9])

$$|(w\varphi)(x)| \left| \sum_{x_k > 0} A_k'(x)[f(x_k) - f(x)] \right| \leqslant C||w\varphi f'||. \tag{19}$$

Let $x_k < 0$. Then

$$\begin{aligned} \left| \sum_{x_k < 0} A_k'(x)[f(x) - f(x_k)] \right| \leqslant & \left| \sum_{x_k < 0} A_k'(x)[f(x) - f(x_{[n/2]+1})] \right| \\ & + \left| \sum_{x_k < 0} A_k'(x)\left[-f(x_{[n/2]}) + f(x_{[n/2]+1})\right] \right| \\ & + \left| \sum_{x_k < 0} A_k'(x)\left[f(x_{[n/2]}) - f(x_k)\right] \right| \\ := {} & \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned} \tag{20}$$

Now Σ_1 and Σ_3 can be estimated as in (18). Hence it remains to estimate Σ_2. Now if $0 < x \leqslant x_{[n/2]+1}$, then we prove that for $\alpha\beta \geqslant 1$

$$\left|w(x)f(x_{[n/2]+1})\right| \leqslant \frac{C}{n}\{||wf|| + ||w\varphi f'||\}. \tag{21}$$

Indeed

$$w(x)|f(x_{[n/2]+1})| \leqslant w(x)\left|f\left(\frac{1}{2}\right) - \int_{x_{[n/2]+1}}^{1/2} f'(y)\,\mathrm{d}y\right|$$
$$\leqslant C||wf||\frac{1}{n^{\alpha\beta}} + C||w\varphi f'||w(x)\left[C + \frac{1}{n^{-\alpha\beta+1}}\right]$$
$$\leqslant \frac{C}{n}[||wf|| + ||w\varphi f'||],$$

i.e., (21) holds true. Moreover, working as usual (see, e.g., [6,9])

$$\varphi(x)\left|\sum_{x_k<0} A_k'(x)\right| \leqslant Cn. \tag{22}$$

Hence by (21) and (22) if $0<x\leqslant x_{[n/2]+1}$ and $\alpha\beta\geqslant 1$

$$\Sigma_2 \leqslant C\{||wf|| + ||w\varphi f'||\}. \tag{23}$$

If $x>x_{[n/2]+1}$, then let x_j denote the closest knot to x. Then by (21) working as usual (see, e.g., [6,9])

$$\Sigma_2 \leqslant C\frac{w(x)}{w\left(x_{[n/2]+1}\right)}w\left(x_{[n/2]+1}\right)\left|f\left(x_{[n/2]+1}\right)\right|\varphi(x)\left|\sum_{x_k<0} A_k'(x)\right|$$
$$\leqslant \frac{C}{n}\{||wf|| + ||wf'\varphi||\}(2j-n)^{\alpha\beta}\varphi(x)\left|\sum_{x_k<0} A_k'(x)\right|$$
$$\leqslant C\{||wf|| + ||wf'\varphi||\}\frac{(2j-n)^{\alpha\beta}}{(j-n/2+1)^{s-1}}$$
$$\leqslant C\{||wf|| + ||wf'\varphi||\}, \tag{24}$$

if $s\geqslant\alpha\beta+1$.

Finally by (19), (20), (23) and (24), the assertion follows. □

Now we introduce the weighted K-functional

$$K^{\varphi}(f;t)_w = \inf_{||w\varphi g'||<\infty}\{||w[f-g]|| + t||w\varphi g'||\}. \tag{25}$$

We have the following

Lemma 8. *Let $f\in C_w$ and let $\omega^{\varphi}(f;t)_w$ and $K^{\varphi}(f;t)_w$ be as in* (7) *and* (25), *respectively. If $\alpha\beta\geqslant 1$, then the following equivalence*

$$\omega^{\varphi}(f;t)_w + \frac{||wf||}{n} \leqslant C\left\{K^{\varphi}(f;t)_w + \frac{||wf||}{n}\right\} \leqslant \bar{C}\left\{\omega^{\varphi}(f;t)_w + \frac{||wf||}{n}\right\}, \tag{26}$$

holds for $0<t<t_0$, where t_0 and the constants in (26) *are independent of f and n.*

Proof. We follow [4,5,11]. Let $t>0$, $M=\min\{k\in\mathbb{N}:k\geqslant t^{-1}\}$ and $t_k=x_{k,M+1}, k=0,\ldots,M+1$, with $x_{k,M+1}$ defined by (3). With $\tau_k=(t_k+t_{k+1})/2$ we define $\psi_k(x)=\psi((x-\tau_k)/\Delta\tau_k)$, where $\psi\in C^\infty(\mathbb{R})$ is a non-decreasing function such that

$$\psi(x)=\begin{cases}1, & \text{if } x\geqslant 1,\\ 0, & \text{if } x\leqslant 0.\end{cases}$$

Recalling the definition of the Steklov function [11]

$$f_\tau(x):=\int_0^1 f(x+\tau u)\,\mathrm{d}u,$$

where $-1<\tau<1$, we introduce the following functions

$$F_{h,k}(x):=\frac{2}{h}\int_{h/2}^{h} f_{\tau\varphi(t_k)}(x)\,\mathrm{d}\tau$$

and

$$G_t(x)=\sum_{k=1}^{M}F_{t,k}(x)\psi_{k-1}(x)(1-\psi_k(x)),$$

with $\psi_0(x)=1$ and $\psi_M(x)=0$. Denoting by I_t the interval $[-1,-t^\beta]\cup[t^\beta,1]$ and proceeding as in [11, pp. 14–16] (cf. also [5] or [4]), it results

$$\|w[f-G_t]\|_{I_t}\leqslant C\Omega^\varphi(f;t)_w, \tag{27}$$

and

$$t\|w\varphi G_t'\|_{I_t}\leqslant C\Omega^\varphi(f;t)_w, \tag{28}$$

with $\Omega^\varphi(f)_w$ given by (5). Now let $J_t=[-t^\beta,t^\beta]$ and let $P_0\in\mathbb{R}$ such that

$$\|w[f-P_0]\|_{J_t}\leqslant 2\inf_{a\in\mathbb{R}}\|w[f-a]\|_{J_t}. \tag{29}$$

We consider the following function

$$\Gamma_t=G_t(1-\Psi_2)+P_0\Psi_2(1-\Psi_4)+G_t\Psi_4, \tag{30}$$

where $\Psi_i(x)=\psi((x-z_i)/\Delta z_i),\ i=2,4,\ z_2=-(2t)^\beta,\ z_3=-t^\beta,\ z_4=t^\beta,\ z_5=(2t)^\beta$ and $\Delta z_i=z_{i+1}-z_i$.

Our aim is to prove that

$$K^\varphi(f;t)_w\leqslant\|w[f-\Gamma_t]\|+t\|\Gamma_t'\varphi w\|\leqslant C\omega^\varphi(f;t)_w. \tag{31}$$

Since

$$\Gamma_t=\begin{cases}G_t, & \text{in } [-1,z_2]\cup[z_5,1],\\ P_0, & \text{in } [z_3,z_4],\\ G_t(1-\Psi_2)+P_0\Psi_2, & \text{in } Z:=[z_2,z_3],\\ P_0(1-\Psi_4)+G_t\Psi_4, & \text{in } \bar{Z}:=[z_4,z_5]\end{cases}$$

and $f=f(1-\Psi_i)+f\Psi_i$, $i=2,4$, we have

$$\|w[f-\Gamma_t]\|\leqslant C\Big\{\|w[f-G_t]\|_{I_t}+\|w[f-P_0]\|_{J_{2t}}\Big\}. \tag{32}$$

Hence by (27), (30) and (32)

$$||w(f-\Gamma_t)|| \leqslant C\left\{\Omega^{\varphi}(f;t)_w + \inf_{a\in\mathbb{R}} ||w[f-a]||_{J_{2t}}\right\} \leqslant C\omega^{\varphi}(f;t)_w. \tag{33}$$

Now we estimate $||w\varphi\Gamma_t'||$. From (30) it follows that

$$||w\varphi\Gamma_t'|| \leqslant ||w\varphi G_t'||_{I_t} + ||w\varphi\Gamma_t'||_{Z\cup\bar{Z}}. \tag{34}$$

Assume $x\in Z$ (the case $x\in\bar{Z}$ is similar). Then

$$\begin{aligned}|\Gamma_t'(x)| &\leqslant |\Psi_2'(x)(G_t-P_0)(x)| + |\Psi_2(x)G_t'(x)|\\ &\leqslant C\Delta z_2^{-1}|(G_t-P_0)(x)| + |G_t'(x)|.\end{aligned}$$

Hence

$$||w\varphi\Gamma_t'||_Z \leqslant C\Delta z_2^{-1}\varphi(x)||w[G_t-P_0]||_Z + ||w\varphi G_t'||_Z.$$

Since $\Delta z_2 \sim t\varphi(x)$, we obtain

$$\begin{aligned}t||w\varphi\Gamma_t'||_Z &\leqslant C\{||w[G_t-P_0]||_Z + t||w\varphi G_t'||_Z\}\\ &\leqslant C\{||w[G_t-f]||_Z + ||w[f-P_0]||_Z + t||w\varphi G_t'||_Z\}.\end{aligned} \tag{35}$$

Thus by (34), (35), (27), (28) and (29)

$$t||w\varphi\Gamma_t'|| \leqslant C\left\{\Omega^{\varphi}(f;t)_w + \inf_{a\in\mathbb{R}} ||w[f-a]||_{J_{2t}}\right\}. \tag{36}$$

Finally (33) and (36) imply (31) and hence the right hand inequality in (26). Now we prove the converse inequality. Working as in [5, Proof of Proposition 2.1, Step 3] (see also [4]) we have

$$\Omega^{\varphi}(f;t)_w \leqslant C\{||w[f-g_t]|| + t||w\varphi g_t'||\}, \tag{37}$$

where $g_t\in AC((-1,1))$ is chosen such that

$$||w(f-g_t)|| + t||w\varphi g_t'|| \leqslant 2K^{\varphi}(f;t)_w. \tag{38}$$

Moreover

$$\begin{aligned}\inf_{a\in\mathbb{R}} ||w[f-a]||_{J_t} &\leqslant ||w[f-g_t]||_{J_t} + \inf_{a\in\mathbb{R}} ||w[g_t-a]||_{J_t}\\ &\leqslant ||w[f-g_t]||_{J_t} + ||wg_t||_{J_t}.\end{aligned} \tag{39}$$

Now if $x\in J_t$, $x\neq 0$, then working as in (21), $\alpha\beta\geqslant 1$,

$$w(x)|g_t(x)| \leqslant Ct\{||wg_t|| + ||w\varphi g_t'||\}. \tag{40}$$

Hence from (40) and (39)

$$\inf_{a\in\mathbb{R}} ||w[f-a]||_{J_t} \leqslant C\{||w[f-g_t]|| + t||w\varphi g_t'||\} + t||wf||. \tag{41}$$

Therefore by (7), (37), (38) and (41)

$$\omega^{\varphi}(f;t)_w \leqslant C\{K^{\varphi}(f;t)_w + t||wf||\},$$

which proves the left-hand inequality in (26). □

Remark 9. From Lemmas 6, 7 and 8 we obtain that $\forall f \in C_w$ and $\forall h \in C_w$ s.t. $||w\varphi h'|| < \infty$

$$\begin{aligned}||w\varphi S_n'(f)|| &\leqslant ||w\varphi S_n'(f-h)|| + ||w\varphi S_n'(h)|| \\ &\leqslant Cn||w[f-h]|| + C||w\varphi h'|| + C||wh|| \\ &\leqslant Cn\omega^{\varphi}\left(f;\frac{1}{n}\right)_w + C||wf||. \end{aligned} \tag{42}$$

Proof of Theorem 2. Because of the interpolatory behaviour of S_n we can assume $x \neq x_k$, $k = 0, \ldots, n$. Let $x > 0$ (the case $x < 0$ is similar). We distinguish two cases.

Case 1: $0 < x < x_{[n/2]+1}$.

Then

$$\begin{aligned} w(x)|f(x) - S_n(f;x)| &\leqslant \frac{w(x)\left\{\sum_{x_k>0} + \sum_{x_k<0}\right\}|x-x_k|^{-s}|f(x)-f(x_k)|}{\sum_{i=0}^n |x-x_i|^{-s}} \\ &:= A + B. \end{aligned} \tag{43}$$

Now

$$\begin{aligned} A &\leqslant w(x)\left|f(x) - f(x_{[n/2]+1})\right| \\ &\quad + \frac{w(x)\sum_{x_k > x_{[n/2]+1}}|x-x_k|^{-s}|f(x_{[n/2]+1}) - f(x_k)|}{\sum_{i=0}^n |x-x_i|^{-s}} \\ &:= A_1 + A_2. \end{aligned} \tag{44}$$

From (44) it follows that

$$A_1 \leqslant 2 \inf_{a \in \mathbb{R}} ||w[f-a]||_{[-1/n^{\beta}, 1/n^{\beta}]}. \tag{45}$$

On the other hand if $x > x_{[n/2]+1}$ by (8)

$$\begin{aligned} w(x)\left|f\left(x_{[n/2]+1}\right) - f(x_k)\right| &= \frac{w(x)}{w\left(\left(x_{[n/2]+1}+x_k\right)/2\right)} w((x_{[n/2]+1}+x_k)/2) \\ &\quad \left|f\left(x_{[n/2]+1}\right) - f(x_k)\right| \\ &\leqslant Cw\left(\frac{x_{[n/2]+1}+x_k}{2}\right)\left|f\left(x_{[n/2]+1}\right) - f(x_k)\right| \\ &\leqslant C\Omega^{\varphi}\left(f; \frac{\left|x_{[n/2]+1} - x_k\right|}{\varphi\left(\left(x_{[n/2]+1}+x_k\right)/2\right)}\right)_w \\ &\leqslant C\left(1 + \frac{n\left|x_k - x_{[n/2]+1}\right|}{\varphi\left(\left(x_{[n/2]+1}+x_k\right)/2\right)}\right)\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w. \end{aligned} \tag{46}$$

Since $\varphi\big(\big(x_{[n/2]+1}+x_k\big)/2\big)>C\varphi\big(x_{[n/2]+1}\big)$ and $\big|x_k-x_{[n/2]+1}\big|<|x-x_k|$, by (44) and (46) we obtain

$$A_2\leqslant C\Omega^\varphi\left(f;\frac{1}{n}\right)_w\frac{\sum_{x_k>x_{[n/2]+1}}|x-x_k|^{-s+1}}{\sum_{i=0}^n|x-x_i|^{-s}}\frac{n}{\varphi\big(x_{[n/2]+1}\big)}. \tag{47}$$

Since (see, e.g., [8])

$$|x-x_k|\geqslant C\frac{\varphi(x)}{n}|k-j|,\ \ k\neq j, \tag{48}$$

and

$$\frac{1}{\sum_{i=0}^n|x-x_i|^{-s}}\leqslant|x-x_j|^s, \tag{49}$$

with x_j the closest knot to x, it follows that

$$\begin{aligned}A_2&\leqslant C\Omega^\varphi\left(f;\frac{1}{n}\right)_w\big|x-x_{[n/2]+1}\big|^s\sum_{k=[n/2]+2}^{n}\big|x_k-x_{[n/2]+1}\big|^{-s+1}\frac{n}{\varphi\big(x_{[n/2]+1}\big)}\\&\leqslant C\Omega^\varphi\left(f;\frac{1}{n}\right)_w\frac{\varphi^s\big(x_{[n/2]+1}\big)}{n^s}\sum_{k=1}^{[n/2]}\frac{n^s}{\varphi^s\big(x_{[n/2]+1}\big)k^{s-1}}\\&\leqslant C\Omega^\varphi\left(f;\frac{1}{n}\right)_w.\end{aligned} \tag{50}$$

Hence from (44), (45) and (50)

$$A\leqslant C\omega^\varphi\left(f;\frac{1}{n}\right)_w. \tag{51}$$

Now we estimate B. Indeed

$$\begin{aligned}B&\leqslant w(x)\big|f(x)-f\big(x_{[n/2]}\big)\big|\\&\quad+w(x)\frac{\sum_{x_k<x_{[n/2]}}\big|f(x_k)-f\big(x_{[n/2]}\big)\big||x-x_k|^{-s}}{\sum_{i=0}^n|x-x_i|^{-s}}\\&=B_1+B_2.\end{aligned} \tag{52}$$

Working as in the estimate of (45)

$$B_1\leqslant 2\inf_{a\in\mathbb{R}}||w[f-a]||_{[-1/n^\beta,1/n^\beta]}. \tag{53}$$

On the other hand working as in (46)–(50)

$$B_2 \leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w \frac{\sum_{x_k<x_{[n/2]}} |x_k - x_{[n/2]}|^{-s+1} n/\varphi\left(x_{[n/2]}\right)}{\sum_{i=0}^{n} |x - x_i|^{-s}}$$
$$\leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w \left|x - x_{[n/2]+1}\right|^s \sum_{x_k<x_{[n/2]}} \frac{n}{\varphi\left(x_{[n/2]}\right)} \left|x_{[n/2]} - x_k\right|^{-s+1}$$
$$\leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w. \tag{54}$$

Finally from (43), (51)–(54)

$$w(x)|f(x) - S_n(f;x)| \leqslant C\omega^{\varphi}\left(f;\frac{1}{n}\right)_w.$$

Case 2: $x_{[n/2]+1} < x$.

Since (see, e.g., [8])

$$|x - x_j| \leqslant C\frac{\varphi(x)}{n}, \tag{55}$$

with x_j the closest knot to x, then

$$w(x)|f(x) - f(x_j)| \leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w. \tag{56}$$

Now we estimate

$$\Sigma := w(x)\frac{\sum_{k\neq j}|f(x) - f(x_k)||x - x_k|^{-s}}{\sum_{i=0}^{n}|x - x_i|^{-s}}$$
$$= w(x)\frac{\{\sum_{\substack{x_k>0\\ k\neq j}} + \sum_{x_k<0}\}|f(x) - f(x_k)||x - x_k|^{-s}}{\sum_{i=0}^{n}|x - x_i|^{-s}}$$
$$:= T_1 + T_2. \tag{57}$$

First we estimate T_1. Working as above, by $x_k > 0$,

$$w(x)|f(x) - f(x_k)| \leqslant C\frac{w(x)}{w((x + x_k)/2)}\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w \left(\frac{n|x - x_k|}{\varphi((x + x_k/2))} + 1\right)$$
$$\leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w \left(\frac{n|x - x_k|}{\varphi(x)} + 1\right).$$

Hence by (48), (49) and (55)

$$T_1 \leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w \frac{n}{\varphi(x)} \sum_{x_k>0} |x - x_k|^{-s+1} \frac{\varphi^s(x)}{n^s} \leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w. \tag{58}$$

Now we estimate T_2. Indeed

$$T_2 \leqslant w(x) \frac{\sum_{x_k<0}\{|f(x)-f(x_{[\frac{n}{2}]+1})|+|f(x_{[\frac{n}{2}]+1})-f(x_k)|\}|x-x_k|^{-s}}{\sum_{i=0}^{n}|x-x_i|^{-s}}$$
$$\coloneqq T_3+T_4, \tag{59}$$

with

$$T_4 \leqslant w(x) \frac{\sum_{x_k<0}[|f(x_k)-f(x_{[\frac{n}{2}]})|+|f(x_{[\frac{n}{2}]+1})-f(x_{[\frac{n}{2}]})|]|x-x_k|^{-s}}{\sum_{i=0}^{n}|x-x_i|^{-s}}$$
$$\coloneqq T_5+T_6.$$

Working as above

$$\begin{aligned} T_3 &\leqslant Cw(x)\sum_{x_k<0}\frac{\Omega^{\varphi}(f;1/n)_w}{w\big((x_{[n/2]+1}+x)/2\big)}\frac{\dfrac{n\big|x-x_{[n/2]+1}\big|}{\varphi\big((x+x_{[n/2]+1})/2\big)}|x-x_k|^{-s}}{\sum_{i=0}^{n}|x-x_i|^{-s}} \\ &\leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w \frac{n}{\varphi(x)}\sum_{x_k<0}|x-x_k|^{-s+1}|x-x_j|^{-s} \\ &\leqslant C\Omega^{\varphi}\left(f;\frac{1}{n}\right)_w. \end{aligned} \tag{60}$$

On the other hand by (45), (48), (49) and (55)

$$\begin{aligned} T_6 &\leqslant \frac{w(x)}{w(x_{[n/2]})}\frac{\sum_{x_k<0}w(x_{[n/2]})\big|f(x_{[n/2]+1})-f(x_{[n/2]})\big||x-x_k|^{-s}}{\sum_{i=0}^{n}|x-x_i|^{-s}} \\ &\leqslant C\frac{w(x)}{w\big(x_{[n/2]}\big)}\inf_{a\in\mathbb{R}}||w[f-a]||_{[-\frac{1}{n^\beta},\frac{1}{n^\beta}]}\sum_{x_k<0}|x-x_k|^{-s}\varphi^s(x)n^{-s} \\ &\leqslant C\inf_{a\in\mathbb{R}}||w[f-a]||_{[-\frac{1}{n^\beta},\frac{1}{n^\beta}]}\left(j-\frac{n}{2}\right)^{\alpha\beta}\left(j-\frac{n}{2}+1\right)^{-s+1} \\ &\leqslant C\inf_{a\in\mathbb{R}}||w[f-a]||_{[-\frac{1}{n^\beta},\frac{1}{n^\beta}]}, \end{aligned} \tag{61}$$

if $s \geqslant \alpha\beta + 1$. Finally we estimate T_5. Working as above

$$T_5 \leqslant Cw(x) \frac{\sum_{x_k<0} \frac{w((x_k + x_{[n/2]})/2)}{w((x_k + x_{[n/2]})/2)} \frac{|f(x_k) - f(x_{[n/2]})|}{|x - x_k|^s}}{\sum_{i=0}^{n} |x - x_i|^{-s}}$$

$$\leqslant C \frac{w(x)}{w(x_{[n/2]})} \Omega^{\varphi}\left(f; \frac{1}{n}\right)_w \frac{\sum_{x_k<0} \frac{n|x_k - x_{[n/2]}|}{|x - x_k|^s \varphi(x_{[n/2]})}}{\sum_{i=0}^{n} |x - x_i|^{-s}}$$

$$\leqslant C\Omega^{\varphi}\left(f; \frac{1}{n}\right)_w \left(j - \frac{n}{2}\right)^{\alpha\beta} \frac{n}{\varphi(x_{[n/2]})} \sum_{x_k<0} |x - x_k|^{-s+1} |x - x_j|^s$$

$$\leqslant C\Omega^{\varphi}\left(f; \frac{1}{n}\right)_w \left(j - \frac{n}{2}\right)^{\alpha\beta} \frac{\varphi(x)}{\varphi(x_{[n/2]})} \left(j - \frac{n}{2} + 1\right)^{-s+2}$$

$$\leqslant C\Omega^{\varphi}\left(f; \frac{1}{n}\right)_w \left(j - \frac{n}{2}\right)^{\alpha\beta+\beta-1} \left(j - \frac{n}{2} + 1\right)^{-s+2}$$

$$\leqslant C\Omega^{\varphi}\left(f; \frac{1}{n}\right)_w, \tag{62}$$

if $s \geqslant \alpha\beta + \beta + 1$. Finally from (56)–(62), if $s \geqslant \alpha\beta + \beta + 1$,

$$w(x)|f(x) - S_n(f; x)| \leqslant C\omega^{\varphi}\left(f; \frac{1}{n}\right)_w.$$

So (9) follows. Now we prove (10). By Lemma 8, (42) and (9)

$$\omega^{\varphi}\left(f; \frac{1}{n}\right)_w + \frac{||wf||}{n} \leqslant CK^{\varphi}\left(f; \frac{1}{n}\right)_w + C\frac{||wf||}{n}$$
$$\leqslant C||w[f - S_n(f)]|| + \frac{C}{n}||w\varphi S_n'(f)|| + C\frac{||wf||}{n}$$
$$\leqslant C\omega^{\varphi}\left(f; \frac{1}{n}\right)_w + C\frac{||wf||}{n},$$

i.e., (10) holds true.

Now we prove (11). From (9) it follows that if $\omega^{\varphi}(f; 1/n)_w = O(t^{\sigma})$, then $||w[f - S_n(f)]|| = O(n^{-\sigma})$. To prove the converse implication, we observe that by Lemmas 5–7 if $||w\varphi g'|| < \infty$,

$$K^{\varphi}\left(f; \frac{1}{n}\right)_w + \frac{||wf||}{n} \leqslant ||w[f - S_k(f)]|| + \frac{1}{n}||w\varphi S_k'(f - g)|| + \frac{1}{n}||w\varphi S_k'(g)|| + \frac{||wf||}{n}$$
$$\leqslant ||w[f - S_k(f)]|| + C\frac{k}{n}||w[f - g]|| + \frac{C}{n}\{||w\varphi g'|| + ||w[f - g]|| + ||wf||\}$$
$$\leqslant ||w[f - S_k(f)]|| + C\frac{k}{n}K^{\varphi}\left(f; \frac{1}{k}\right)_w + \frac{C}{n}||wf||.$$

Hence if $||w[f - S_n(f)]|| = O(k^{-\sigma})$, then by [11, Lemma 9.3.4, p. 122]

$$K^{\varphi}\left(f;\frac{1}{n}\right)_w + \frac{||wf||}{n} \leqslant \frac{C}{n^{\sigma}}.$$

Consequently by Lemma 8 it follows that $\omega^{\varphi}(f;1/n)_w = O(n^{-\sigma})$, i.e., (11) holds true. Finally working as in [11, Corollary 7.3, p. 86] by Lemma 7 and (9) we deduce (12).

Proof of Proposition 3. Letting $f(x) = |x|^{\gamma-\alpha}$, with $1/\beta \leqslant \gamma < 1$, we show that (13) will give us a contradiction. Indeed from (13) we get

$$|x|^{\alpha}|[|x|^{\gamma-\alpha} - p_n(x)]| \leqslant \frac{C}{n}, \quad \forall |x| \leqslant 1,$$

with p_n an algebraic polynomial (even) of degree at most n. Hence

$$\frac{1}{n^{\alpha}}|[|x|^{\gamma-\alpha} - p_n(x)]| \leqslant \frac{C}{n}, \ \frac{1}{n(1+1/n^2)^{1/2}} \leqslant |x| \leqslant 1. \tag{63}$$

Now making in (63) the change of variable

$$y = \left(\frac{x^2 + 1/n^2}{1 + 1/n^2}\right)^{1/2}, \quad x \in [-1, 1],$$

we get, for $|x| \leqslant 1$ and $0 < \delta < 1$,

$$\left| n^{-\alpha+1-\delta}\left(\frac{x^2 + 1/n^2}{1 + 1/n^2}\right)^{(\gamma-\alpha)/2} - q_n(x)\right| \leqslant \frac{C}{n^{\delta}}, \quad \forall |x| \leqslant 1, \quad 0 < \delta < 1, \tag{64}$$

with q_n a suitable (even degree) algebraic polynomial. Setting

$$f_n(x) = n^{-\alpha+1-\delta}\left(\frac{x^2 + 1/n^2}{1 + 1/n^2}\right)^{(\gamma-\alpha)/2},$$

we deduce from (64) that $f_n(x) \in \mathrm{Lip}\delta$ in $[-1/2, 1/2]$. Hence in particular

$$\left| f_n\left(\frac{1}{n}\right) - f_n\left(\frac{2}{n}\right)\right| \leqslant \frac{C}{n^{\delta}},$$

which implies

$$\frac{1}{n}\left| f_n'\left(\frac{d}{n}\right)\right| \leqslant \frac{C}{n^{\delta}}, \quad 1 < d < 2. \tag{65}$$

But a direct computation shows that

$$\frac{1}{n}\left| f_n'\left(\frac{d}{n}\right)\right| \sim \frac{1}{n} n^{-\alpha+1-\delta}\left(\frac{1}{n^2}\right)^{\frac{\gamma-\alpha}{2}-1}\frac{1}{n} \sim n^{-\delta-\gamma+1},$$

which contradicts (65) for $\gamma < 1$. And the assertion follows.

Proof of Theorem 4. Because of the interpolatory behaviour of operator $\bar{S}_n$, we assume $x \neq y_k, k = 0, \ldots, n$. Let $x > 0$ (the case $x < 0$ is similar). We distinguish four cases.

Case 1: $0 < x < y_0$.

Then, Letting

$$\Sigma := w(x)\frac{\sum_{k\neq 0,n}|x-y_k|^{-s}|f(y_k)-f(y_0)|}{\sum_{i=0}^{n}|x-y_i|^{-s}},$$

$$\begin{aligned} w(x)|f(x)-\overline{S}_n(f;x)| &\leqslant w(x)|f(x)-f(y_0)| + w(x)\frac{\sum_{k=1}^{n}\frac{|f(y_k)-f(y_0)|}{|x-y_k|^{s}}}{\sum_{i=0}^{n}|x-y_i|^{-s}} \\ &\leqslant w(x)|f(x)| + \frac{w(x)}{w(y_0)}|(wf)(y_0)| + w(x)|f(y_n)-f(y_0)| + \Sigma \\ &\leqslant C\left\{\epsilon_f(x) + \frac{w(x)}{w(y_0)}\epsilon_f(y_0) + \Sigma\right\}. \end{aligned} \tag{66}$$

Now we prove that

$$\Sigma \leqslant C\left\{\frac{w(x)}{w(y_0)}\epsilon_f(\mu_n) + \frac{w(x)}{w(\mu_n)}\epsilon_f(1)\right\},$$

with $\mu_n = Cn^{-\delta}, 0 < \delta < 1$. Indeed

$$\Sigma = w(x)\left\{\sum_{|y_k|>\mu_n} + \sum_{|y_k|\leqslant\mu_n}\right\}\frac{|x-y_k|^{-s}|f(y_k)-f(y_0)|}{\sum_{i=0}^{n}|x-y_i|^{-s}} := \Sigma_1 + \Sigma_2. \tag{67}$$

Now

$$\begin{aligned} \Sigma_2 &\leqslant w(x)|f(y_0)| + w(x)\sum_{|y_k|\leqslant\mu_n}\frac{|x-y_k|^{-s}|f(y_k)|}{\sum_{i=0}^{n}|x-y_i|^{-s}} \\ &\leqslant \frac{w(x)}{w(y_0)}\epsilon_f(y_0) + \frac{w(x)}{w(y_0)}\epsilon_f(\mu_n). \end{aligned}$$

On the other hand

$$\Sigma_1 \leqslant \frac{w(x)}{w(y_0)}\epsilon_f(y_0) + \frac{w(x)}{w(\mu_n)}\epsilon_f(1).$$

Hence by (66) and (67) if $|x| < x_0$

$$w(x)|f(x)-\overline{S}_n(f;x)| \leqslant C\{\epsilon_f(x) + \frac{w(x)}{w(y_0)}\epsilon_f(\mu_n) + \frac{w(x)}{w(\mu_n)}\epsilon_f(1)\}.$$

Case 2: $x > y_0$ and $y_k > 0$.

Let y_j denote the closest knot to x. Since $|x-y_j|\leqslant C\frac{\varphi(x)}{n}$ (see e.g. [8]), then

$$w(x)|f(x)-f(y_j)| = \frac{w(x)}{w(\frac{x+y_j}{2})}w\left(\frac{x+y_j}{2}\right)|f(x)-f(y_j)|\leqslant Cw\left(\frac{x+y_j}{2}\right)|f(x)-f(y_j)|$$
$$\leqslant C\Omega(f;|x-y_j|)_w \leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w.$$

Similarly

$$w(x)|f(x)-f(y_{j-1})|\leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w.$$

Moreover working as usually (see e.g. [8])

$$w(x)\sum_{\substack{k\neq j,j-1\\ y_k>0}}\frac{|x-y_k|^{-s}|f(x)-f(y_k)|}{\sum_{i=0}^n|x-y_i|^{-s}}\leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w.$$

Case 3: $\dfrac{|y_k+x|}{2}<y_0,\ x>y_0,\ y_k<0.$

Then

$$w(x)|f(x)-f(y_k)|\leqslant \epsilon_f(x)+\frac{w(x)}{w(y_k)}\epsilon_f(y_k)\leqslant \epsilon_f(x)+\frac{w(x)}{w(y_0)}\epsilon_f(2y_0+x). \tag{68}$$

Moreover

$$\sum_{\frac{|x+y_k|}{2}<y_0}\frac{|x-y_k|^{-s}}{\sum_{i=0}^n|x-y_i|^{-s}}\leqslant \sum_{-2y_0-x<y_k<0}\frac{|x-y_k|^{-s}}{\sum_{i=0}^n|x-y_i|^{-s}}\leqslant |x-y_j|^s\sum_{-2y_0-x<y_k<0}|x-y_k|^{-s}$$
$$\leqslant C\frac{\varphi(x)^s}{n^s}\sum_{y_k<0}|x-y_k|^{-s}\leqslant C\frac{\varphi^s(x)}{n^s}\frac{n}{x^s}$$
$$\leqslant \frac{C}{n^{s-1}x^{s/\beta}}.$$

Hence

$$\frac{w(x)}{w(y_0)}\sum_{\frac{|x+y_k|}{2}<y_0}\frac{|x-y_k|^{-s}}{\sum_{i=0}^n|x-y_i|^{-s}}\leqslant\frac{C}{n^{s-1-\alpha}x^{\frac{s}{\beta}-\alpha}}\leqslant\frac{C}{n^{s-1-\alpha}}n^{\frac{s}{\beta}-\alpha}\leqslant\frac{C}{n^{s(1-\frac{1}{\beta})-1}}, \tag{69}$$

if $s>\alpha\beta$ and $s>1+\frac{1}{\beta-1}$. Therefore by (68) and (69)

$$w(x)\sum_{|x+y_k|/2<y_0}\frac{|f(x)-f(y_k)||x-y_k|^{-s}}{\sum_{k=0}^n|x-y_k|^{-s}}\leqslant C\left\{\epsilon_f(x)+\frac{\epsilon_f(2y_0+x)}{n^{s(1-\frac{1}{\beta})-1}}\right\}.$$

Case 4: $\dfrac{|x+y_k|}{2}>y_0$ and $y_k<0, x>y_0.$

Then working as above

$$
\begin{aligned}
&\frac{w(x)}{w(y_0)} \sum_{y_k < -2y_0 - x} \frac{w(\frac{x+y_k}{2})|f(x)-f(y_k)||x-y_k|^{-s}}{\sum_{i=0}^{n}|x-y_i|^{-s}} \\
&\leqslant C\frac{w(x)}{w(y_0)}\Omega\left(f;\frac{\varphi(x)}{n}\right)_w \frac{n}{\varphi(x)}\frac{\varphi^s(x)}{n^s} \sum_{y_k<-2y_0-x} |x-y_k|^{-s+1} \\
&\leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w x^\alpha \frac{n^\alpha}{n^{s-1}}\varphi^{s-1}(x) \sum_{y_k<-2y_0-x} |x-y_k|^{-s+1} \\
&\leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w \frac{x^\alpha x^{(s-1)(1-\frac{1}{\beta})}}{n^{s-1-\alpha}} \frac{n}{x^{s-1}} \leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w \frac{x^{\alpha-(s-1)/\beta}}{n^{s-2-\alpha}} \\
&\leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w n^{-s+2+\alpha} n^{-\alpha+(s-1)/\beta} \leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w n^{-s(1-1/\beta)+2-1/\beta} \\
&\leqslant C\Omega\left(f;\frac{\varphi(x)}{n}\right)_w
\end{aligned}
$$

if $s > \alpha\beta + 1$ and $s \geqslant 1 + \frac{\beta}{\beta-1}$. Finally from Cases 1–4 the assertion follows.

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